South East Asian J. of Math. & Math. Sci. **ISSN : 0972-7752 (Print)**
Vol. 14, No. 3 (2018), pp. 49-56

ODD GRACEFULNESS OF TREES OF DIAMETER FOUR WITH CYCLE

J. Jeba Jesintha, R. Jaya Glory* and K. Farzana

PG Department of Mathematics,
Women's Christian College, Chennai, INDIA
E-mail: jjesintha_75@yahoo.com, farzana12hussain@gmail.com

*Department of Mathematics,
Anna Adarsh College, Chennai, INDIA
E-mail: godblessglory@gmail.com

(Received: May 8, 2018)

Abstract: In 1991, Gnanajothi [3] introduced a labeling method called *odd graceful labeling.* A graph G with q edges is said to be odd graceful if there is an injection f from $V(G) \rightarrow \{0, 1, 2, ..., (2q-1)\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are $1, 3, 5, ..., (2q-1)$. In this paper, we prove the odd gracefulness on trees of diameter four with cycle.

Keywords and Phrases: Odd graceful labeling, Cycle, Tress of Diameter four.

2010 Mathematics Subject Classification: 05C78.

1. Introduction and Definition

Rosa [5], in 1967, introduced the first graph labeling method called graceful labeling. A graceful labeling of a graph G with q edges and vertex set V is an injection $f : V(G) \rightarrow \{0, 1, 2, ..., q\}$ with the property that the resulting edge labels are also distinct, where an edge incident with vertices u and v is assigned the label $|f(u) - f(v)|$. In 1991, Gananajothi [3] introduced *odd graceful labeling.* An odd graceful labeling is an injection f from $V(G) \rightarrow \{0, 1, 2, ..., (2q-1)\}$ such that, when each edge xy is assigned the label $|f(x) - f(y)|$, the resulting edge labels are $1, 3, 5, ..., (2q-1)$. Lekha [4] proved the following results on cycle related graphs: Joint sum of two copies of C_n of even order, joining two copies of C_n of even order by a path, two copies of even cycles C_n sharing a common edge is odd graceful.Gnanajothi [3] proposed the conjecture, All trees are odd graceful. She also proved this conjecture for all trees with order up to 10. Christian Barrientos

[1] has verified this conjecture to all trees of order up to 12. For an exhaustive survey on odd graceful labeling refer to dynamic survey by Gallian [2].

Graph labeling is an active area of research in graph theory which has rigorous applications in coding theory, communication networks, optimal circuits layouts and graph decomposition problems.

In this paper, we define a new graph called tree attachment with cycle and we prove that the tree attachment with cycle is odd graceful.

Definition 1.1 *The graph G obtained by identifying the root vertex of tree T^i, where $1 \leq i \leq m$,with the vertices of cycle C_m is called the tree attachment with cycle. In other words, the graph $G = C_m \odot T^i$.*

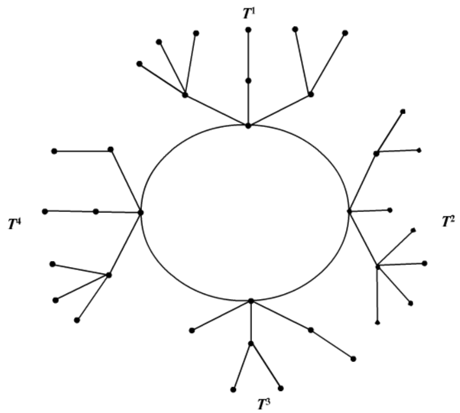

Fig 1: The graph G which is the tree attachment with cycle

2. Main Result

In this section, we prove that the graph G which is obtained by identifying the root vertex T^i where $1 \leq i \leq m$,with the vertices of cycle C_m when $m \equiv 0\ (mod 4)$ is odd graceful.

Theorem 2.1. *The Graph $G = C_m \odot T^i$ when $m \equiv 0\ (mod 4)$ admits odd graceful labeling.*

Proof. Let T be a tree of diameter four with two levels. The root vertex of T at level zero is of degree n and the degree of internal vertices of T at level one depends upon the pendant edges added to it. The pendant vertices of T at level two are of degree one. We describe the graph G as follows. The vertices in the cycle C_m in G are denoted as $u_1, u_2, u_3, ..., u_m$ in the clockwise direction. Consider non-isomorphic m copies of the tree T. That is, the first copy of T denoted by T^1 is attached at the vertex u_1 of C_m. The second copy of T denoted by T^2 is attached at the vertex u_2 of C_m. In general, the i^{th} copy of T denoted by T^i is attached at the vertex u_i of C_m where $1 \leq i \leq m$ for $m \equiv 0\ (mod 4)$. The resultant graph after attaching the non-isomorphic m copies of T^i of T is represented as G.

In the graph G, the vertices in the level one of tree T^1 at u_1 are denoted as $v_{11}, v_{12}, v_{13}, ..., v_{1n}$. The vertices in the level one of tree T^2 at u_2 are denoted as $v_{21}, v_{22}, v_{23}, ..., v_{2n}$. Similarly the vertices in the level one tree T^3 at u_3 are denoted as $v_{31}, v_{32}, v_{33}, ..., v_{3n}$. In general, the vertices in the level one of tree T^i attached at u_i are denoted as v_{ij} where $1 \leq i \leq m$ and $1 \leq j \leq n$.

In the graph G, the pendant vertices in the level two of the tree T^1 attached at v_{11} are denoted as $w_{1,1,k^1}$ where $1 \leq k \leq a_1^1$, the pendant vertices attached at v_{12} are denoted as $w_{1,2,k^1}$ where $1 \leq k \leq a_2^1$. Similarly,the pendant vertices attached in the level two of the tree T^1 attached at v_{1n} are denoted as w_{1,n,k^1} where $1 \leq k \leq a_n^1$. The pendant vertices in the level two of the tree T^2 attached at v_{21} are denoted as $w_{2,1,k^2}$ where $1 \leq k \leq a_1^2$, the pendant vertices attached at v_{22} are denoted as $w_{2,2,k^2}$ where $1 \leq k \leq a_2^2$. Similarly,the pendant vertices attached in the level two of the tree T^2 attached at v_{2n} are denoted as w_{2,n,k^2} where $1 \leq k \leq a_n^2$. The pendant vertices in the level two of the tree T^3 attached at v_{31} are denoted as $w_{3,1,k^3}$ where $1 \leq k \leq a_1^3$, the pendant vertices attached at v_{32} are denoted as $w_{3,2,k^3}$ where $1 \leq k \leq a_2^3$. Similarly, the pendant vertices attached in the level two of the tree T^3 attached at v_{3n} are denoted as w_{3,n,k^3} where $1 \leq k \leq a_n^3$. In general, the vertices in the level two of tree T^i attached v_{ij} are denoted as w_{i,j,k^i} where $1 \leq i \leq m$ and $1 \leq j \leq n$, $1 \leq k \leq a_t^i$,$1 \leq t \leq n$.

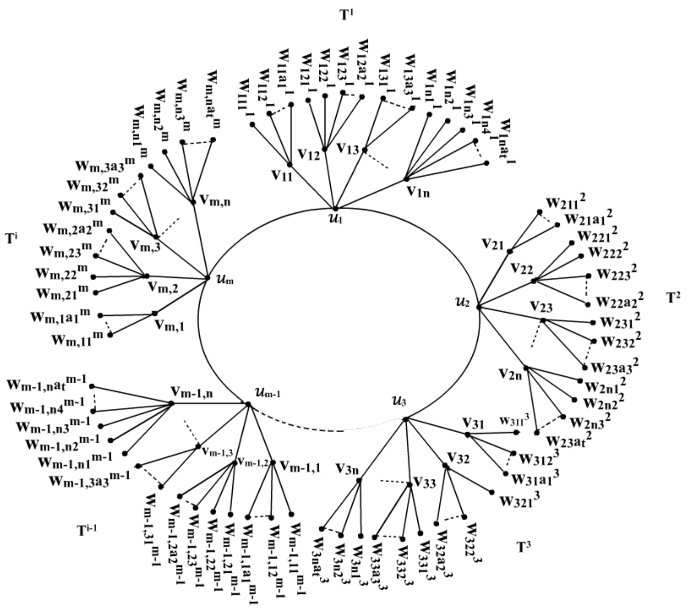

Figure 2: The graph G

Let N denote the total number of pendant vertices at level two. In the graph G, the total number of pendant vertices at level two of T^1 is denoted by N_1. The total number of pendant vertices at level two of T^2 is denoted by N_2. The total number of pendant vertices at level two of T^3 is denoted by N_3. In general, the total number of pendant vertices at level two of T^i is denoted by N_r where $1 \leq r \leq m$, $1 \leq i \leq m$ and $m \equiv 0\ (mod 4)$.

Let $|V(G)| = p$ and $|E(G)| = q$. The Graph G has $p = (n+1)m + \Sigma N_r$ vertices and $q = (n+1)m + \Sigma N_r$ edges where $1 \leq r \leq m$.

The vertices of the Graph G when $m \equiv 0\ (mod 4)$ is labeled by first labeling the vertices of a cycle C_m and then labeling the other vertices of Graph G .
The **vertex labels** for the cycle C_m given below.

$$\begin{array}{llll} f(u_{2i-1}) & = (2q-1)-(i-1)(2n+2) & for & 1 \leq i \leq \frac{m}{4} \\ f(u_{2i-1}) & = (2q-3)-(i-1)(2n+2) & for & \frac{m}{4}+1 \leq i \leq \frac{m}{2} \\ f(u_{2i}) & = 2n+(i-1)(2n+2) & for & 1 \leq i \leq \frac{m}{2} \end{array} \tag{2.1}$$

The **vertex labels** for level 1 of trees T^i at u_i is defined as follows

$$f(v_{2i-1,j}) = (i-1)(2n+2)+(2j-2) \quad for \quad 1 \leq i \leq (\tfrac{m}{2}),\ \ 1 \leq j \leq n$$

$$f(v_{2i,j}) = \begin{cases} (2q-3)-(i-1)(2n+2)-(2j-2), \\ \quad for \quad 1 \leq i \leq (\frac{m}{4}),\ \ 1 \leq j \leq n \\ \\ (2q-5)-(i-1)(2n+2)-(2j-2) \\ \quad for \quad \frac{m}{4}+1 \leq i \leq (\frac{m}{2}),\ \ 1 \leq j \leq n \end{cases} \tag{2.2}$$

The **vertex labels** for level 2 of trees T^i at u_i is defined as follows

$$\begin{aligned} f(w_{(2i-1)}, j, k^{(2i-1)}) = {} & 2i(n+1)+(2k+2j-2n-5)+2\sum_{t=0}^{j-1}(a_t{}^{(2i-1)}) \\ & +2\sum_{r=0}^{2i-2}(N_r) \\ & for \quad 1 \leq i \leq (\tfrac{m}{2}),\ 1 \leq k \leq a_t{}^{2i-1},\ 1 \leq j \leq n,\ 1 \leq t \leq n \end{aligned}$$

$$f(w_{(2i)}, j, k^{(2i)}) = \begin{cases} (2q-2)-2\sum_{t=0}^{j-1}(a_t{}^{(2i)})-2\sum_{r=1}^{2i-1}(N_r)-(2k+2j+2i)- \\ 2n(i-1)+4 \\ for \ \ 1 \leq i \leq (\frac{m}{4}),\ 1 \leq k \leq a_t{}^{2i},\ 1 \leq j \leq n,\ 1 \leq t \leq n \\ \\ (2q-2)-2\sum_{t=0}^{j-1}(a_t{}^{(2i)})-2\sum_{r=1}^{2i-1}(N_r)-(2k+2j+2i)- \\ 2n(i-1)+2 \\ for \ \ (\frac{m}{4}+1) \leq i \leq (\frac{m}{2}),\ 1 \leq k \leq a_t{}^{2i},\ 1 \leq j \leq n,\ 1 \leq t \leq n \end{cases} \tag{2.3}$$

From the equations (2.1) to (2.3) we see that the vertex labels for the graph G are distinct.

Now we compute the **edge labels** for the graph G by first computing the edge labels for cycle C_m for $m \equiv 0\ (mod 4)$ as follows:

$$|f(u_{2i-1}) - f(u_{2i})| = \begin{cases} (2q-2n-1) - 2(i-1)(2n+2) \\ \qquad for\ 1 \leq i \leq (\frac{m}{4}) \\ \\ (2q-2n-3) - 2(i-1)(2n+2) \\ \qquad for\ (\frac{m}{4}+1) \leq i \leq (\frac{m}{2}) \end{cases}$$

$$|f(u_{2i+1}) - f(u_{2i})| = \begin{cases} (2q-4n-3) - 2(i-1)(2n+2), \\ \qquad for\ 1 \leq i \leq (\frac{m}{4}-1) \\ \\ (2q-4n-5) - 2(i-1)(2n+2), \\ \qquad for (\frac{m}{4}) \leq i \leq (\frac{m}{2}-1) \end{cases}$$

$$|f(u_1) - f(u_m)| \quad = 2q - m(n+1) + 1, \tag{2.4}$$

The **edge labels** for edges incident with vertices of level 1 of trees T^i at u_i are defined as follows

$$|f(u_{2i-1}) - f(v_{(2i-1),j})| = \begin{cases} (2q-1) - 2(i-1)(2n+2) - (2j-2), \\ \qquad for \quad 1 \leq i \leq (\frac{m}{4}), \quad 1 \leq j \leq n \\ \\ (2q-3) - 2(i-1)(2n+2) - (2j-2), \\ \qquad for \quad (\frac{m}{4}+1) \leq i \leq (\frac{m}{2}), \quad 1 \leq j \leq n \end{cases}$$

$$|f(u_{2i}) - f(v_{2i,j})| \quad = \begin{cases} (2q-3-2n) - 2(i-1)(2n+2) - (2j-2) \\ \qquad for \quad 1 \leq i \leq (\frac{m}{4}), \quad 1 \leq j \leq n \\ \\ (2q-5-2n) - 2(i-1)(2n+2) - (2j-2) \\ \qquad for \quad (\frac{m}{4}+1) \leq i \leq (\frac{m}{2}), \quad 1 \leq j \leq n \end{cases} \tag{2.5}$$

The edge labels for edges incident with vertices of level 2 of trees T^i at u_i are defined as follows

$$|f(v_{2i-1,j}) - f(w_{(2i-1)}, j, k^{(2i-1)})| = 2\sum_{t=0}^{j-1}(a_t{}^{(2i-1)}) + 2\sum_{r=0}^{2i-2}(N_r) + (2k-1)$$

$$for \quad 1 \le i \le (\tfrac{m}{2}),\ 1 \le k \le a_t{}^{2i-1},\ 1 \le j \le n,\ 1 \le t \le n$$

$$|f(v_{2i,j}) - f(w_{(2i)}, j, k^{(2i)})| \qquad = 1 + 2\sum_{t=0}^{j-1}(a_t{}^{(2i)}) + \sum_{r=1}^{2i-1}(N_r) + (2k-2)$$

$$for \quad 1 \le i \le (\tfrac{m}{2}),\ 1 \le k \le a_t{}^{2i},\ 1 \le j \le n,\ 1 \le t \le n \tag{2.6}$$

From the above computed edge labels we see that the edge labels for the graph G are the distinct odd numbers from the set $1, 3, 5, ..., (2q-1)$.

Hence the for the graph G is odd graceful.

We illustrate the above mentioned theorem in the figure 3.

3. Illustration

When $m = 4,\ n = 3,\ p = 36,\ q = 36,\ a_0^i = N_0 = 0, \quad N_1 = 6,\ N_2 = 6,\ N_3 = 3,\ N_4 = 5$

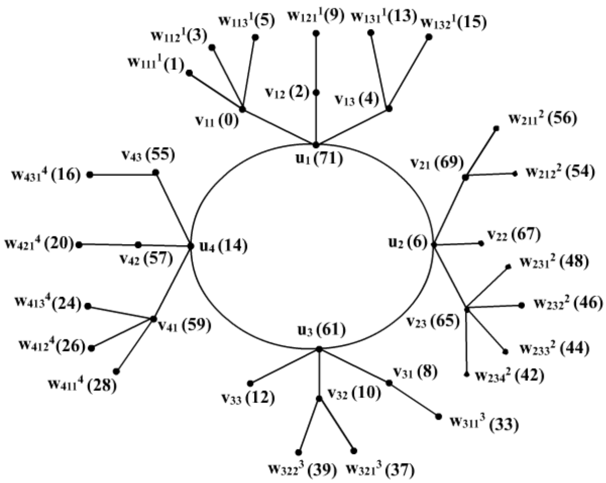

Figure 3 : An odd graceful labeling of the graph G

Corollary 3.1. *The $m-$ isomorphic copies of trees of diameter four with arbitrary root degree attached at each vertex of a cycle C_m where $m \equiv 0\ (mod 4)$ admits odd graceful labeling.*

Proof. From the Theorem, if we let root vertex of T at level zero to be of degree n, the internal vertices of T at level one to be of degree three and the pendant vertices of T at level to be of degree one then we get the odd graceful labeling of

$m-$ isomorphic copies of trees of diameter four with arbitrary root degree attached at each vertex of a cycle C_m.

References

[1] Christian Barrientos, Odd-Graceful Labelings of Trees of Diameter 5, *AKCE J. Graphs, Combin.,* Vol. 6-No. 2, (2009), 307-313.

[2] Gallian. J. A., A dynamic survey of graph labeling, *J Combin.* (2018).

[3] Gnanajothi. R B., *Topics in Graph Theory,* Ph.D. Thesis, Madurai Kamaraj University, 1991.

[4] Lekha B, *Some interesting results in the theory of Graphs*, Thesis PhD, Saurashtra University.

[5] Rosa. A., On certain valuations of the vertices of a graph, *Theory of Graphs (Internat Symposium, Rome, July 1966), Gordon and Breach, N.Y. and Dunod Paris*, (1967), 349-355.